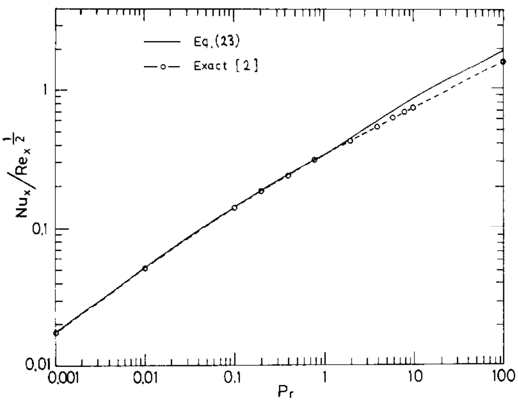

FIG. 2. Comparison of the exact solution and equation (23) for the case of a flat plate, $\beta = 0$.

start to diverge for larger Prandtl numbers and, finally, become parallel to each other at a sufficiently large Prandtl number. For $Pr \leqslant 1$, the maximum error of 70% by the conventional asymptote was reduced to 13% by the two-region model [1] and it is further reduced to only 0.8% by the present theory. The accuracy is dramatically improved, particularly in the range of $0.1 \leqslant Pr \leqslant 1$. It should also be noticed that the present theory converges faster than the other two asymptotes at a small Prandtl number.

Calculations also indicate that the present theory shifts from underestimation to overestimation between Prandtl numbers of 0.8 and 1.0, which implies that there is one point between these two values at which the present theory and the exact solution yield identical values. Since equation (23) is an asymptote for small Prandtl numbers, as indicated by (24), it also matches the exact solution asymptotically at zero Prandtl number. Therefore, the present theory matches the exact solution at two points. On the other hand, the two-region model and conventional asymptote both match the exact solution asymptotically at zero Prandtl number, but deviate ever afterwards, with the two-region model always underestimating and the convectional asymptote overestimating. In the sense of the number of points matching the exact solution, the two-region model and the conventional asymptote are of first-order, and the present theory is a second-order asymptote.

For an accelerating flow, $\beta > 0$, the present theory also yields better predictions and converges faster than the other two asymptotes. For instance, the maximum error for the case of $\beta = 2$ for $Pr \leqslant 1$ is 2.5%, compared to 5.6% by the two-region model [1] and 32% by the conventional asymptote. For a decelerating flow when the diverging angle is not too large, the present theory again gives better predictions. For $Pr \leqslant 1$, the maximum error is 3.3% for $\beta = -0.1$, compared to 15% by the two-region model, and 8.4% for $\beta = -0.16$, compared to 16% [1]. However, for a strong decelerating flow, $\beta = -0.198838$, the present theory yields results with about the same accuracy as the two-region model up to $Pr = 0.1$, but becomes less accurate for larger Prandtl numbers. At $Pr = 1$, the error is 26% compared to 15% for the previous model.

It is interesting to see that the present theory has its best prediction for the case of a flat plate. This can be explained by the following argument.

Since the heat transfer rate depends strongly on the velocity field close to the wall, a good velocity simulation in this region is essential to the heat transfer rate prediction. Under the non-slip and impermeable conditions, the velocity boundary-layer equation on the wall becomes

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)\bigg|_{y=0} = -u_\infty \frac{\mathrm{d}u_\infty}{\mathrm{d}x}. \tag{26}$$

For the special case of a flat plate, $\mathrm{d}u_\infty/\mathrm{d}x = 0$, the second derivative is identical to zero. This indicates that the velocity profile close to the wall can be well approximated by a linear profile. Incidentally, as a linear profile is assumed in the velocity boundary layer by the present theory, there is good reason to believe that the present theory is at its best for the case of a flat plate.

REFERENCES

1. Y.-M. Chen, Heat transfer of a laminar flow passing a wedge at small Prandtl number: a new approach, *Int. J. Heat Mass Transfer* **28**, 1517–1523 (1985).
2. H. L. Evans, Mass transfer through laminar boundary layers 7. Further similar solutions to be the *b*-equation for the case $\beta = 0$. *Int. J. Heat Mass Transfer* **5**, 35–57 (1962).

Int. J. Heat Mass Transfer. Vol. 29, No. 3, pp. 492–495, 1986
Printed in Great Britain
0017-9310/86 $3.00 + 0.00

Double diffusion from a horizontal line source in an infinite porous medium

STEVEN E. LARSON† and DIMOS POULIKAKOS‡

† Engine Electronics Automotive and Industrial Electronics Group, Motorola, Inc., 1299 E. Algonquin Road, Schaumburg, IL 60196, U.S.A.

‡ Department of Mechanical Engineering, University of Illinois at Chicago, P.O. Box 4348, Chicago, IL 60680, U.S.A.

(*Received* 13 *May* 1985 *and in final form* 6 *September* 1985)

1. INTRODUCTION

THE PRESENT paper aims to analyze an important fundamental problem in porous media natural convection: the phenomenon of time-dependent heat, mass and fluid flow induced by a horizontal line source producing simultaneously heat and a chemical species. The study will determine the effect of the presence of the chemical species on the main features of the flow field which, after it originates in the vicinity of the line source, penetrates the unbounded porous surroundings. In addition to its fundamental nature, the present problem finds practical applications illustrated by the spreading of chemical pollutants generated by exothermic reactions in the earth's crust, the chemical industry, the disposal of nuclear wastes and the natural convection cooling of buried electrical cables.

Previous studies of natural convection from a line source

NOMENCLATURE

A dimensionless parameter, $(\beta_c mk)/(\beta qD)$
B dimensionless parameter, $(\lambda Le)^{1/2}$
c species concentration [kg m^{-3}]
c_p specific heat of fluid at constant pressure
D mass diffusivity [m^2 s^{-1}]
d distance between wall and source
$Ei(\eta^2)$ exponential integral, $-\int_{\eta^2}^{\infty}[\exp(-\xi)/\xi]\,d\xi$
g gravitational acceleration
K permeability of porous medium
Le Lewis number, α/D
m rate of species generation from the line source [kg s^{-1}]
P pressure
q rate of heat generation from the line source [W]
r radial coordinate
Ra Darcy-modified Rayleigh number based on the permeability of the porous matrix and the heat generation rate at the source, $K^{3/2}g\beta q/\mu\alpha k$
S function defined in equation (16)
T temperature
t time
u radial velocity
v tangential velocity
x horizontal Cartesian coordinate
y vertical Cartesian coordinate.

Greek symbols
α effective thermal diffusivity of porous medium
β coefficient of thermal expansion
β_c coefficient of concentration expansion
γ Euler's constant, 0.5772
η similarity variable, $r/2t^{1/2}$
θ angle
λ dimensionless parameter, ϕ/σ
μ viscosity
ν kinematic viscosity
ξ dummy variable
ρ fluid density
σ heat capacity ratio, $\sigma = [\phi(\rho c_p)_f + (1-\phi)(\rho c_p)_s]/(\rho c_p)_f$
τ dimensionless time, $t_*\alpha/\sigma d^2$
ϕ porosity of porous matrix
ψ streamfunction, $u_* = r^{-1*}\,\partial\psi_*/\partial\theta$, $v_* = -\partial\psi_*/\partial r_*$

Subscripts
$*$ dimensional quantity
∞ reference state.

in an unbounded porous medium driven only by thermal buoyancy are exemplified by refs. [1–4]. Reference [1] deals with the high Rayleigh number regime, refs. [2, 3] with the low Rayleigh number regime and ref. [4] reports numerical calculations valid in a wide Rayleigh number range. The present investigation relies on asymptotic expansions in the Rayleigh number [2, 5, 6] to obtain the transient flow, temperature and concentration fields. The results reported in this paper extent the findings in ref. [2], for they account for the presence of concentration gradients in the driving buoyancy mechanism and depict the effect of mass transfer on the flow and temperature fields.

2. MATHEMATICAL FORMULATION

The system of interest is a horizontal line source located in an unbounded porous medium. Initially, the temperature and the concentration are uniform everywhere in the system. Suddenly, the line source begins to liberate heat (per unit depth) at a rate of q W and a chemical substance at a rate m kg s^{-1}. The density of the generated substance is different from the density of the fluid saturating the porous medium. The dimensionless governing equations according to the Darcy flow model [7] are

$$r\frac{\partial^2\psi}{\partial r^2}+\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2} = Ra\left[\left(\sin\theta\frac{\partial T}{\partial\theta}-r\cos\theta\frac{\partial T}{\partial r}\right) - A\left(\sin\theta\frac{\partial c}{\partial\theta}-r\cos\theta\frac{\partial c}{\partial r}\right)\right] \quad (1)$$

$$\frac{\partial T}{\partial t}+\frac{1}{r}\left[\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right] = \frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} \quad (2)$$

$$\lambda\frac{\partial c}{\partial t}+\frac{1}{r}\left[\frac{\partial\psi}{\partial\theta}\frac{\partial c}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial c}{\partial\theta}\right] = \frac{1}{Le}\left[\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{1}{r^2}\frac{\partial^2 c}{\partial\theta^2}\right]. \quad (3)$$

The non-dimensionalization of the above equations was based on the following definitions

$$t=\frac{t_*}{\sigma K/\alpha},\quad r=\frac{r_*}{K^{1/2}},\quad \psi=\frac{\psi_*}{\alpha},$$
$$(u,v)=\frac{(u_*,v_*)}{\alpha/K^{1/2}},\quad c=\frac{c_*-c_\infty}{m/D},\quad T=\frac{T_*-T_\infty}{q/k}. \quad (4)$$

All the parameters appearing in the governing equations are defined in the Nomenclature. The fluid is modeled as Boussinesq incompressible whereby the density is assumed to be constant everywhere except in the buoyancy terms in (1) where it is given by

$$\rho = \rho_\infty[1-\beta(T_*-T_\infty)+\beta_c(c_*-c_\infty)]. \quad (5)$$

The boundary and initial conditions necessary to complete the mathematical formulation are

$$u_*\to 0,\quad v_*\to 0,\quad T_*\to T_\infty,$$
$$c_*\to c_\infty \quad \text{as} \quad r_*\to\infty \quad \text{or} \quad t_*=0$$
$$v_*=\frac{\partial u_*}{\partial\theta}=\frac{\partial T_*}{\partial\theta}=\frac{\partial c_*}{\partial\theta}=0 \quad \text{at} \quad \theta=\pm\pi/2$$
$$\lim_{r_*\to 0}\left[-k(2\pi r_*)\frac{\partial T_*}{\partial r_*}\right]=q,\quad \lim_{r_*\to 0}\left[-D(2\pi r_*)\frac{\partial c_*}{\partial r_*}\right]=m. \quad (6)$$

It is worth stressing that parameter A in equation (1) measures the relative contribution of the concentration and the thermal buoyancy in driving the flow. Negative values of A imply that the two diffusion mechanisms drive the flow jointly whereas positive values of A indicate that the two mechanisms oppose each other.

3. PERTURBATION SOLUTION FOR THE TRANSIENT STATE

When the Rayleigh number is small, the problem outlined in the previous sections accepts an approximate theoretical solution of the form

$$(\psi, T, c) = \sum_{n=0}^{\infty} (\psi_n, T_n, c_n)Ra^n \qquad (7)$$

where ψ_n, T_n and c_n depend on r, θ and t. To obtain ψ_n, T_n, c_n we substitute equation (7) into the governing equations (1)–(3), collect terms of equal order in Ra and solve the resulting differential equations. The method of solution of these equations is identical to that outlined in refs. [2, 5, 6]. Here, we present only the final expressions for the temperature, concentration and flow fields. The intermediate steps, similar to those in [2, 5, 6], are outlined in ref. [8].

The zeroth-order solutions ($n = 0$) correspond to the state of pure diffusion, i.e. in the absence of fluid motion. Hence, we set $\psi_0 = 0$ and according to Carslaw and Jaeger [9]

$$T_0 = -\frac{1}{4\pi} Ei(-\eta^2), \quad c_0 = -\frac{1}{4\pi} Ei[-(B\eta)^2] \qquad (8, 9)$$

where

$$\eta = \frac{r}{2t^{1/2}}, \quad B = (\lambda Le)^{1/2}. \qquad (10, 11)$$

The first-order solutions read

$$\psi_1 = \frac{t^{1/2}}{4\pi} \cos\theta \left\{ \frac{\exp(-\eta^2) - 1}{\eta} + \eta\, Ei(-\eta^2) + \frac{A}{B}\left(\frac{1 - \exp[-(B\eta)^2]}{B\eta} - B\eta\, Ei[-(B\eta)^2] \right) \right\} \qquad (12)$$

$$T_1 = \frac{t^{1/2}\sin\theta}{16\pi^2} \left\{ (\ln\eta)[(\gamma-2)\eta - \eta^3] + \eta(\ln\eta)^2 + \eta\frac{2-\gamma}{2} + \eta^3\frac{3-\gamma}{2} + \cdots + \frac{A}{B}\left[B(\eta^3+\eta)\ln(B\eta) - 2B\eta\ln\eta\ln(B\eta) + B(\eta - \gamma\eta)\ln\eta + B\eta\frac{\gamma-2}{2} + B\eta^3\frac{\gamma-3}{2} + B\eta(\ln\eta)^2 + \ldots \right] \right\} \qquad (13)$$

$$c_1 = \frac{t^{1/2}\sin\theta\, Le}{16\pi^2 B} \left\{ (\ln B\eta)[(\gamma-2)B\eta - (B\eta)^3 + B\eta\ln\eta - B\eta\ln B] + (\ln B)[B\eta + (B\eta)^3] + B\eta\frac{2-\gamma}{2} + (B\eta)^3\frac{3-\gamma}{2} + \cdots + A\left[\ln(B\eta)((2-\gamma)B\eta + (B\eta)^3) - B\eta\ln^2(B\eta) + B\eta\frac{\gamma-2}{2} + (B\eta)^3\frac{\gamma-3}{2} + \ldots \right] \right\}. \qquad (14)$$

Parameter B has a paramount effect on the flow field in the case where the two buoyancy mechanisms are opposing each other ($A > 0$). For example, a map of streamlines, $\psi_1/(t^{1/2}/2\pi) = \text{const}$, is shown in Fig. 1, for $A = 0.25$, $B = 0.1$. Figure 1 indicates that values of B of order less than unity create a downward flow far from the source, engulfing the toroidal vortex near the source. To shed light on the implications of this result the physical meaning of parameter B needs to be examined. This parameter represents the ratio of two length scales in the transient regime: the thermal penetration length scale, $l_T \sim O(\sqrt{\alpha t/\sigma})$ divided by the species penetration length scale, $l_c \sim O(\sqrt{Dt/\phi})$. Values of B less than unity indicate that the species generated by the source diffuses faster than heat at early times. Therefore, outside the region of extent l_T in which the heating effect of the source is felt, the species concentration gradients act alone initiating the downward flow shown in Fig. 1. The outer boundary of the region in which the heating effect of the source is not negligible is defined by the streamline $\psi_1/(t^{1/2}/2\pi) = 0$ drawn by using a dashed line in Fig. 1. It is

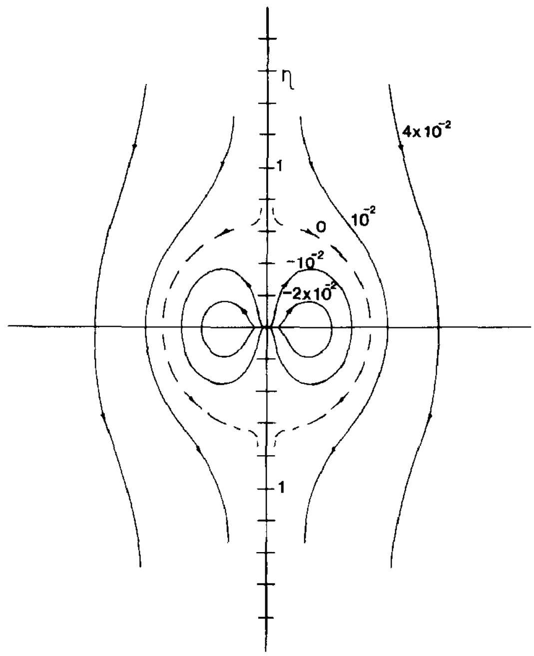

FIG. 1. The effect of parameter B on the flow field for $B = 0.1$, $A = 0.25$. On each streamline, $\psi_1/(t^{1/2}/2\pi) = \text{const}$.

worth noting that for the special case $B = 1$ the streamline pattern $\psi_1/t^{1/2}(1-A) = \text{const}$ [equation (12)] consists of a toroidal vortex identical to that reported in ref. [2] for $A = 0$. The effect of A on the radial distribution of c_1 and T_1 is shown in Fig. 2. Negative values of A amplify the impact of the source on the temperature and concentration distributions while positive values of A weaken that effect relative to the $A = 0$ limit (no species generation). Finally, important conclusions relevant to the steady-state flow field may be drawn by examining the behavior of the two velocity components at large times ($\eta \to 0$)

$$(u, v) = -\frac{1}{4\pi}[\ln\eta - A\ln(B\eta)](\sin\theta, \ \cos\theta). \qquad (15)$$

Clearly, as $t \to \infty$ the magnitude of the velocity grows without bound. Hence, no steady state exists for the problem of low Ra

FIG. 2. The effect of parameter A on the radial variation of the first-order convective correction or the temperature field $(16\pi^2 T_1/t^{1/2}\sin\theta)$ or the concentration field $(16\pi^2 c_1/t^{1/2}\sin\theta)$ for $B = Le = 1$.

double-diffusive convection from a line source in a porous medium, in agreement with the findings in ref. [2] for natural convection driven by temperature gradients alone.

Due to the prohibitive complexity of the algebraic calculations, higher-order solutions in Ra were impossible to obtain.

4. THE PRESENCE OF A VERTICAL INSULATED WALL IN THE VICINITY OF THE SOURCE

Based on the previous results it is possible to shed light on the effect of the presence of a vertical insulated wall on the flow field induced by the line source. Assume that the insulated vertical wall constitutes the y_*-axis of an $(x_*\text{–}y_*)$ Cartesian coordinate system and that the line source is located at $x_* = d$, $y_* = 0$. This arrangement is equivalent to an arrangement consisting of two line sources positioned at $y_* = 0, x_* = \pm d$, with the vertical wall removed [2]. The zeroth-order solution corresponds to the case of no fluid motion and it is reported in ref. [9]. Hence, it is not repeated here for brevity. As explained in ref. [2], due to the linearity of the momentum equation the solution for ψ_1 is simply the superposition of solutions for line sources at $x = \pm 1, y = 0$. In this part of the study d was used as the reference length for the non-dimensionalization. The final expression for ψ_1 reads

$$\psi_1 = \frac{\tau^{1/2}}{4\pi}(S_+ + S_-) \tag{16}$$

where

$$\begin{aligned} S_\pm = {} & \frac{2\tau^{1/2}(x\pm 1)}{(x\pm 1)^2+y^2}\left\{\exp\left[-\frac{(x\pm 1)^2+y^2}{4\tau}\right]-1\right\} \\ & -\frac{x\pm 1}{2\tau^{1/2}}\int_{[(x\pm 1)^2+y^2]/4\tau}^{\infty}\frac{\exp(-\xi)}{\xi}\,d\xi \\ & -\frac{A}{B}\left\{\frac{2\tau^{1/2}(x\pm 1)}{(x\pm 1)^2+y^2}\,\frac{1}{B}\left(\exp\left[-B^2\frac{(x\pm 1)^2+y^2}{4\tau}\right]-1\right)\right. \\ & \left. -\frac{B(x\pm 1)}{2\tau^{1/2}}\int_{B^2[(x\pm 1)^2+y^2]/4\tau}^{\infty}\frac{\exp(-\xi)}{\xi}\,d\xi\right\}. \end{aligned} \tag{17}$$

The first-order equations for c_1 and T_1 are non-linear, therefore, it is not possible to obtain c_1 and T_1 by superposition. The streamline pattern $\psi_1/(1-A) = \text{const}$ for $\tau = 1, B = 1$ was identical to the streamline pattern in ref. [2] where the concentration-gradient-induced buoyancy was neglected ($A = 0$). Basically, the presence of the wall flattens the streamlines in the wall vicinity. For an illustration of this effect ref. [2] is recommended.

Acknowledgement—This research was supported in part by the National Science Foundation under grant no. ENG-8451144.

REFERENCES

1. R. A. Wooding, Convection in a saturated porous medium at large Rayleigh or Péclet number, *J. Fluid Mech.* **15**, 527–544 (1963).
2. D. A. Nield and S. P. White, Natural convection in a infinite porous medium by a line heat source, *Mathematics and Models in Engineering Science, DSIR*, New Zealand, pp. 121–128 (1982).
3. C. E. Hickox, Thermal convection at low Rayleigh number from concentrated sources in porous media, *J. Heat Transfer* **103**, 232–236 (1981).
4. C. E. Hickox and H. A. Watts, Steady thermal convection from a concentrated source in a porous medium, *J. Heat Transfer* **102**, 248–253 (1980).
5. D. Poulikakos, On buoyancy induced heat and mass transfer from a concentrated source in an infinite porous medium, *Int. J. Heat Mass Transfer* **28**, 621–629 (1985).
6. A. Bejan, Natural convection in an infinite porous medium with a concentrated source, *J. Fluid Mech.* **89**, 97–107 (1978).
7. P. Cheng, Heat transfer in geothermal systems, *Adv. Heat Transfer* **14**, 1–105 (1979).
8. S. E. Larson, Double diffusion from a horizontal line source in an infinite porous medium. M.S. project, Mechanical Engineering Department, University of Illinois at Chicago (June 1983).
9. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Oxford, Oxford University Press (1959).

Int. J. Heat Mass Transfer. Vol. 29, No. 3, pp. 495–499, 1986
Printed in Great Britain
0017-9310/86 \$3.00 + 0.00

A numerical solution to moving boundary problems—application to melting and solidification

MEHMET A. HASTAOGLU
Department of Chemical and Petroleum Engineering, The University of Calgary, Calgary, Alberta, Canada T2N 1N4

(*Received* 14 *December* 1984 *and in final form* 12 *September* 1985)

1. INTRODUCTION

A LARGE number of technically important problems are classified as moving boundary. Heat transfer problems with a phase change, litospheric movement according to plate tectonics, gas–solid reactions occurring in a moving reaction zone are all of the moving boundary type. For various kinds of such problems there are solutions available [1]. For the case that the thermal conductivity varies linearly with temperature Cho and Sunderland [2] presented an exact solution; Voller and Cross [3] have investigated the same problem in two dimensions. Cheung *et al* [4] presented numerical solutions for a finite slab with internal heat generation.

Analytical solutions, although very convenient, can only be applied to very specific cases. In situations where physical properties depend on system variables the analytical solutions are impossible. Problems with various complexities and boundary conditions can be analyzed numerically using superfast computers.

An approximation commonly adopted in numerical approach is that the phase boundary movement and also the changes in transient quantities occur at a constant rate in a